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Torsional Vibration Characteristics of a Semi-Infinite Non-Homogeneous Solid Cylinder

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Abstract:

This paper investigates the free torsional vibration of a non-homogeneous semi-infinite solid circular cylinder with stress-free and rigid boundaries. The spatial variation in rigidity μ and density ρ is defined through specific functional forms $\mu = \mu_0 e^{nz}$, $\rho = \rho_0 e^{nz}$. The governing equations are formulated and solved to determine the vibrational behaviour of the system. Numerical results are presented for various values of the parameter n, illustrating the effects of material non-homogeneity on displacement and stress components under torsional motion.

Key-words: *Torsional vibration, Non-homogeneous solid cylinder, Frequency equation, Boundary condition, Stress components.*

1. Introduction:

The problem of torsional oscillation in a circular cylinder has been investigated by many researchers, including Love [1], Kolsky [2], Davies [3], and others. Forced torsional vibration in a circular cylinder has also been studied by several authors, such as Mitra [4], Banerjee [5], Campbell and Tsao [6], and Mondal [7]. In all these studies, the curved boundary of the cylinder was assumed to be stress-free.

However, in many engineering applications, a protective cover or shield is provided, which constrains the particles on the curved boundary from moving. This boundary condition can be mathematically modelled as a rigid boundary. Recently, free torsional vibration in an isotropic, homogeneous, infinitely long solid circular cylinder with a rigid boundary was considered by Rao [8]. Mondal [9] examined the case where the rigidity (μ) and density (ρ) vary along the axial distance.

In the present section, we extend the previous problem to the case of free torsional vibration of a semi-infinite solid circular cylinder with a rigid boundary, where the material is non-homogeneous in rigidity and density, following an exponential variation along the axial distance. For comparison, we also consider the analogous case in which the curved surface is free from traction. The solution is obtained using the method of separation of variables. Explicit formulae are presented for the stress components and particle displacements, which are then computed at a fixed depth and compared graphically. For the rigid boundary case, the nodal cylinder pattern is found to be the same as that in the stress-free boundary case.

2. Model formulation and solution procedure:

We consider a semi-infinite solid circular cylinder of radius a. The origin is taken at the center of one end of the cylinder, with the z-axis directed along its axis. Using cylindrical polar coordinates (r, θ, z) to specify points within the cylinder, we first study the case in which the curved boundary surface is stress-free. It is assumed that the stress distribution is radially symmetric, so that the displacement is purely tangential and independent of the angular coordinate θ. Under this assumption, the displacement components are given by

$$u_r = u_z = 0, \quad u_\theta = u_\theta(r, z, t) \tag{1}$$

For such a displacement field, the stress components that remain non-zero are:

$$\sigma_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial_r} - \frac{u_\theta}{r}\right), \quad \sigma_{\theta z} = \mu\frac{\partial u_\theta}{\partial z} \tag{2}$$

To account for material non-homogeneity, the modulus of rigidity and density are assumed to have the form

$$\mu = \mu_0 e^{nz}, \quad \rho = \rho_0 e^{nz} \tag{3}$$

Where μ_0, ρ_0 are constants and n is a rational number.

The only non-vanishing equation of motion is

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + n\frac{\partial u_\theta}{\partial z} = \frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2} \tag{4}$$

Where, $\beta^2 = \mu_0/\rho_0$

When the motion of every particle of the body is simple harmonic and of period $2\pi/p$, the displacement may be expressed by

$$u_\theta = \mathrm{v}(r, z) e^{ipt} \tag{5}$$

In the case of free vibration, the equation of motion and boundary condition can be satisfied only if p is a one of the roots of the frequency equation.

Substituting (5) in the equation (4) it takes the form

$$\frac{\partial^2 \mathrm{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \mathrm{v}}{\partial r} - \frac{\mathrm{v}}{r^2} + \frac{\partial^2 \mathrm{v}}{\partial z^2} + n\frac{\partial \mathrm{v}}{\partial z} + \frac{p^2}{\beta^2}\mathrm{v} = 0$$

Using the method of separation of variables, we assume the solution of the equation in the form

$$\mathrm{v} = R(r)\ F(z) \tag{6}$$

Then the above equation reduces to

$$\frac{1}{R}\left[\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2} + \frac{p^2}{\beta^2}R\right] = -\frac{1}{F}\left[\frac{d^2 F}{dz^2} + n\frac{dF}{dz}\right] = -s^2$$

Where s is a separation constant, thus we have

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{1}{r^2}\right)R = 0 \tag{7}$$

Where,

$$\lambda^2 = \frac{p^2}{\beta^2} + s^2 \tag{8}$$

And

$$\frac{d^2 F}{dz^2} + n\frac{dF}{dz} - s^2 F = 0 \tag{9}$$

The solution of (7) is given by

$$R = C\ J_1(\lambda r) + D\,Y_1(\lambda r)$$

Where *C, D* are constant and J_1, Y_1 are Bessel functions of 1st and 2nd kind respectively of order unity. Since for a solid cylinder, μ_θ/r must be finite at $r=0$, so, $D=0$ and we have

$$R = C\, J_1(\lambda r) \tag{10}$$

The solution of (9) is

$$F = e^{-\frac{n}{2}z}\left(C_1 e^{\frac{\sqrt{n^2+4s^2}}{2}z} + D_1 e^{-\frac{\sqrt{n^2+4s^2}}{2}z} \right) \tag{11}$$

Where, C_1, D_1 are constants.

As u_θ must be finite when

$$z \to \infty,$$

So, $C_1=0$ and

$$F = D_1 e^{\frac{-n-\sqrt{n^2+4s^2}}{2}z} \tag{12}$$

From (5), (6), (10), (12) we can write

$$u_\theta = A e^{-\frac{\left(n+\sqrt{n^2+4s^2}\right)}{2}z} J_1(\lambda r) e^{ipt} \tag{13}$$

Where, $A=CD_1$

Againe, putting the value of u_θ and μ in the first relation of (2) we get

$$\sigma_{r\theta} = -A\lambda\mu_0 e^{\frac{n-\sqrt{n^2+4s^2}}{2}z} J_2(\lambda r)\, e^{ipt} \tag{14}$$

If curved surface is free from traction, boundery condition is

$$\sigma_{r\theta} = 0 \quad \text{on} \quad r = a$$

Which, yields the frequency equation

$$J_2(\lambda a) = 0 \tag{15}$$

The first ten positive non-zero roots of the frequency equation (15) Watson [10] are as follows

5.1356233, 8.4172441, 11.6198412, 14.7959518, 17.9598195, 21.1169971, 24.2701123, 27.4205736, 30.5692045, 33.7165195.

The stress component $\sigma_{\theta z}$ is from (3) and (13)

$$\sigma_{\theta z} = -A\mu_0 e^{\frac{n-\sqrt{n^2+4s^2}}{2}z} J_1(\lambda r)\, e^{ipt} \tag{16}$$

Considering the case when the curved surface of the cylinder is rigid. In this case the boundery condition is $u_\theta=0,$ when, $r = a$ which in turn gives

$$J_1(\lambda a) = 0 \tag{17}$$

3.8317060, 7.0155867, 10.1734681, 13.3236919, 16.4706301, 19.6158585, 22.7600844, 25.9036721, 29.0468285, 32.1896799.

It is to be noted that in this case, the roots are smaller than those for the stress-free case.

The expressions for displacement and stresses in this case take the form

$$u_\theta = Ae^{-\frac{\left(n+\sqrt{n^2+4s^2}\right)}{2}z} J_1(\lambda r)e^{ipt} \tag{18}$$

$$\sigma_{r\theta} = -A\lambda\mu_0 e^{\frac{n-\sqrt{n^2+4s^2}}{2}z} J_2(\lambda r)\, e^{ipt} \tag{19}$$

$$\sigma_{\theta z} = -A\mu_0 e^{\frac{n-\sqrt{n^2+4s^2}}{2}z} J_1(\lambda r)\, e^{ipt} \tag{20}$$

3. Particular case:

a) In order to obtain the solution when the material of the cylinder is homogeneous isotropic and the curved surface of the cylinder stress free we substitute n = 0 in the equations (13), (14) and (16) and they reduce to

$$\left.\begin{aligned} u_\theta &= Ae^{-sz}J_1(\lambda r)e^{ipt} \\ \sigma_{r\theta} &= -A\lambda\mu_0 e^{-sz}J_2(\lambda r)e^{ipt} \\ \sigma_{\theta z} &= -A\mu_0 se^{-sz}J_1(\lambda r)\, e^{ipt} \end{aligned}\right\} \tag{21}$$

b) When the material of the cylinder is non-homogeneous, the non-homogeneity being characterized by the equation

$$\mu = \mu_0 e^{nz}, \quad \rho = \rho_0 e^{nz},$$

and the curved surface of the cylinder is free from traction, the solution will be obtained by setting n=2 in the equations (13), (14) and (16). The above type of expressions for the modulus of rigidity and density are taken in view of realistic Earth model B of Bullen [11]. For instance, using the value $k = 0.0005$, $\mu_0 = 0.625$, $\rho_0 = 3.22$ for a depth of 33Km we obtain the density and rigidity by using the above forms in close agreement with the data of Bullen at subsequent depth. Now for this case the displacement and stress components are

$$\left.\begin{aligned} u_\theta &= A e^{-\left(1+\sqrt{1+s^2}\right)z} J_1(\lambda r) e^{ipt} \\ \sigma_{r\theta} &= -A\lambda\mu_0 e^{-\left(1+\sqrt{1+s^2}\right)z} J_2(\lambda r) e^{ipt} \\ \sigma_{\theta z} &= -A\mu_0 s e^{-\left(1+\sqrt{1+s^2}\right)z} J_1(\lambda r)\, e^{ipt} \end{aligned}\right\} \tag{22}$$

4. Numerical Results and Discussions:

To obtain the displacement and stress for first mode of vibration, we take

$$\frac{s}{k} = 1, \text{ ka=1, } \frac{z}{a} = 1, \text{ ka=1, and k= 0.0005}$$

The distribution of $\frac{u_\theta}{A}$ x10^2 and $\frac{u_\theta}{A}$ x10^4 at different radii for n=0, 2 are shown in fig.1 both for rigid and stress-free boundary. The stress components $-\frac{\sigma_{\theta z}}{A\mu_0}$ and $-\frac{\sigma_{r\theta}}{A\mu_0}$ are shown in fig.2 and fig.3 respectively.

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The roots of the frequency equation for stress free boundary are greater than those of the rigid boundary as mentioned earlier. For higher modes of vibration there exist different values of r for which u_θ vanishes. In all the cases the non-homogeneity decreases the values of u_θ, $-\sigma_{r\theta}$ and $-\sigma_{\theta z}$ at every point within the cylinder. The maximum values of the displacement and stresses occur near r/a=0.5 for rigid boundary.

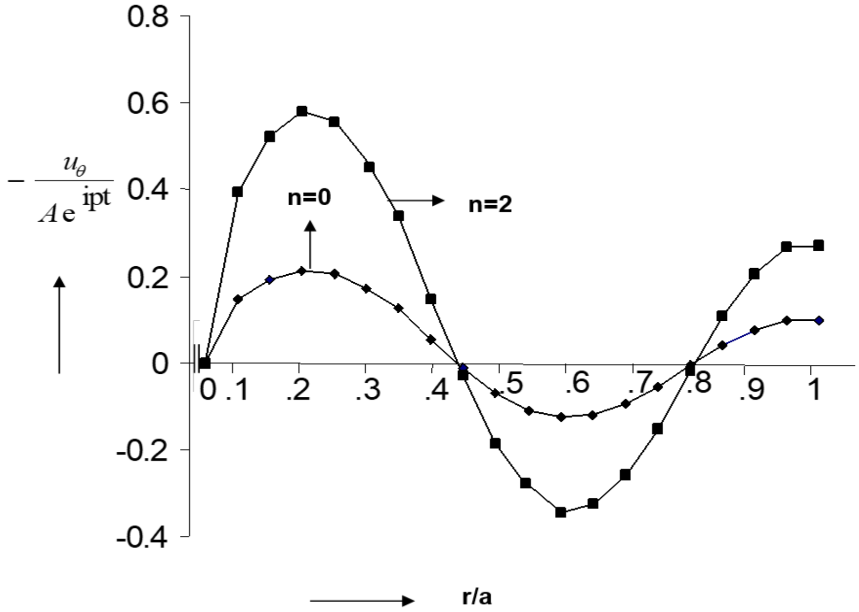

Fig.1: The displacement component **u$_\theta$** along the radius of the cylinder.

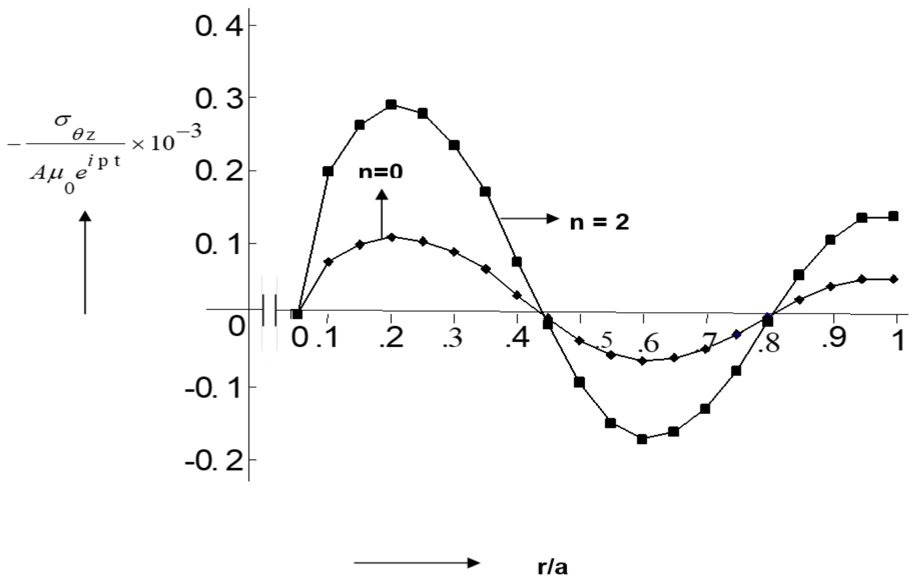

Fig.2: The stress component **σ$_{\theta z}$** along the radius of the cylinder.

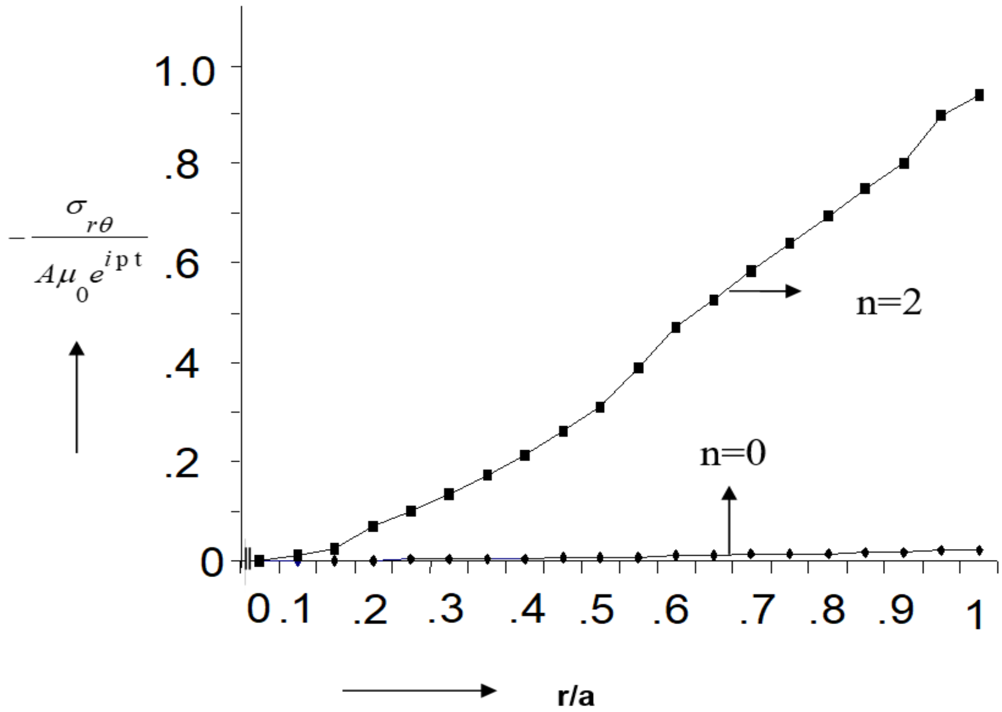

Fig.3. The stress component $\sigma_{r\theta}$ along the radius of the cylinder.

6. Conclusion:

The roots of the frequency equation for stress free boundary are greater than those of the rigid boundary as mentioned earlier. For higher modes of vibration there exist different values of *r* for which u_θ vanishes. In all the cases the non-homogeneity decreases the values of u_θ, $-\sigma_{r\theta}$ and $-\sigma_{rz}$ at every point within the cylinder. The maximum values of the displacement and stress occur near r/a=0.5 for rigid boundary.

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