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ANALYTICAL STUDY OF DIFFERENCE BETWEEN REAL AND COMPLEX DIFFERENTIABLE FUNCTIONS AND THEIR USES

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ABSTRACT

The purpose of this paper is to know interesting difference with real differentiable function f: $\mathbb{R} \rightarrow \mathbb{R}$ and complex differentiable function f: $\mathbb{C} \rightarrow \mathbb{C}$. we could know what fundamental properties makes the difference in them basically, and we present & analysis that complex analytic function strongly support the important results like a)f(z) is analytic at a point implies all possible derivatives of f(z)are analytic at that point. b)f(z) is non -constant analytic function on $\mathbb{C}$ implies it must be constant. c)Maximum value (or Minimum value) attains in boundary of analytic function d)Analytic function is always given by a Power series e)Open mapping theorem f) Schwarz's lemma & stability in Uniform convergence etc. Some counter examples are given to demonstrate above results where complex differentiable functions exhibit "a miraculous amount of structure (properties)!''And "more real" and not so over real differentiable functions thus it is imperative to study carefully Finally in addition we could know the applications of real & complex differentiable functions.

***Keywords*: Analytic Function, Complex Differentiable, Infinite Differentiable, Power Series, Real Differentiable**

I INTRODUCTION

This is well known that real and complex analysis plays a very important role in engineering & a well-studied subject of mathematical analysis often depend on analyzing the behavior of the functions in real valued function (1-D(dimensional) variable) & complex valued function(2-D variable) geometrically perhaps 2-D geometry has more structure than 1-D. Here the real differentiable function to be one that looks, on a small scale, "a *stretch of a line around that area*" where as a complex differentiable function on the plane to actually look like "*a shape preserving transformation"* in a small area which is only change of scale or rotation(regularly) rather than stretch, reflect or change shapes(irregularly) thus the 2-D geometry of complex functions have much more strong restrictive as wide angles of direction to approach to a point and to determine complete characters just by knowing its behavior in small part of the whole(=hole) domain that is why the name *holomorphic function* (*analytic function*). [1], [2], [3]

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From the literature review we may realize that major results in real & complex analysis are differ in rigorous definitions of limits and continuity and derivatives must be understood clearly with real numbers before one can understand analogous definitions with complex no's, in fact that real analysis deals with a restriction of no imaginary part, gives it some special properties that are not true with complex numbers, and thus one would get different results based on that fact. And a lot of real analysis makes critical use of the fact that $\mathbb{R}$ is totally ordered, $\mathbb{C}$ loses order but gains a lot of 'elegant' properties such as well behaved derivatives, completeness, algebraically closured field, multiplication of complex no's, Well known functions in real line become multi-valued in complex plane etc. [1], [2], [4]

II DIFFERENCE BETWEEN THE REAL ANALYSIS CONCEPTS AND COMPLEX ANALYSIS CONCEPTS

The name "complex analysis" is a little deceiving, because the subject in fact analyses only those functions of complex numbers $\mathbb{C}$ that are differentiable at a point, or for all $z \in \mathbb{C}$ or some other open set $\Omega \subseteq \mathbb{C}$

Notes: Complex *analytic functions* is different from complex derivatives, it is defined as any complex function differentiable (in the complex sense) in an open set is analytic. Consequently, in complex analysis, the term *analytic function* is synonymous with *holomorphic function* and we seen now complex analytic functions exhibit properties that do not hold true in general for real analytic functions.

2.1. The concept of a derivative in the complex plane looks apparently the same as a derivative on the real line:

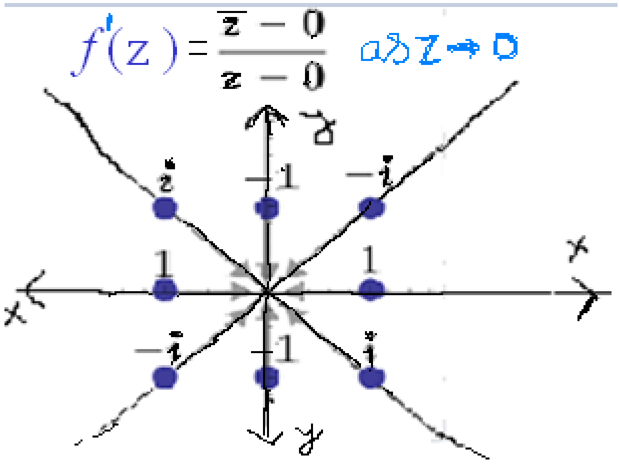
A complex function f has a derivative at z_0 if: $f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$, exist finitely.

For real derivatives at x_0, it looks exactly the same; we usually write "x_0" for "z_0", now *f* is a function of a real variable.

The key difference is that if h is real, it can only approach zero from left and right direction in real line. If h is complex, it can approach zero not only from an infinite number of directions, but also any spiral path etc, in complex plane.

- For example, $f(z) = \bar{z}$ is not complex-differentiable at zero, because as shown below, the value of $\frac{f(z) - f(0)}{z - 0}$ varies depending on the direction from which z approaches zero.

Along the real axis (2 directions), *f* equals the function $g(z) = z$ & the limit is 1,



Whereas along the imaginary axis, *f* equals *h* (*z*) = −*z* & the limit is −1. Other directions give yet different limits. Thus differentiability in the complex plane depends on the existence of a much more restrictive limit.[1]

2.2. The condition of differentiability is so strong in the complex plane that if a function has one derivative, it has all derivatives (analyticity)

This is certainly not true of the real functions (differentiability)

- For example, $f(x) = x^2\sin(1/x)$ on R−{0} and set f(0)=0 .f(x) is 1- time differentiable only but not 2-time in any neighbourhood of x=0.

But the corresponding: $f(z)=z^2\sin(1/z)$ is first only not analytic in any region containing z = 0.so we must not conclude it is infinitely differentiable in any neighbourhood of z =0.(∵singularity at z =0)

2.3. Maximum (or Minimum) modules theorem and principle

It states that if *f* is analytic inside a bounded domain *D*, continuous up to the boundary of *D*, (or and for minimum only, non-zero at all points), then |*f* (z)| takes its maximum value (or minimum value) on the boundary of *D* respectively.

- For example, f(x) = sin(x) is bounded on I= [-π,π] but attains maximum value at some interior point x= π/2 i.e., sin π/2=1 but not on boundary of I

Moreover, sin(x) =0 at x=0, an interior point still it attains minimum value at the boundary point at x = - π & π of |*f* (x)| on I=[-π,π].

Thus, both maximum & minimum modules principles need not hold true in case of real bounded functions

2.4. Open Mapping Theorem

It states if D is connected open subset of ℂ & *f* : D→ℂ is an open map i.e. *f* sends open subset of D to open subsets of ℂ.

- For example: On real line, the differentiable function $f(x) = x^2$ is not an open mapping because image of open interval (-2,2) is the half -open interval [0,4)

Thus, this the core difference between analyticity & real-differentiability.

2.5. Schwarz's lemma

If f is analytic in domain D: $|z|<1$ such that $f(0) = 0$ and $|f(z)| \leq 1 \ \forall \ z \in D \Rightarrow |f(z)| \leq |z|$ and $|f^1(0)| \leq 1$.

- For example: consider real valued function of real variable: $f(x)= \dfrac{2x}{x^2+1}$,in $-1 \leq x \leq 1$

$\ni -1 \leq f(x) \leq 1$, $f^1(x)$ is continuous & $f(0) =0$.

Thus, there exist real functions which satisfies hypothesis but fail to get conclusion as in lemma.

2 .6.Stability of Complex differentiable w.r.t uniform convergence

The limit of derivatives and derivative of limit can be interchangeable for sequence of analytic functions in convergence regions which is not true for real valued function of real variable.

2 .7.Liouville's theorem

The image of an entire, non -constant function must be unbounded.

- For example: cos (z), sin (z) are infinitely complex differentiable in $\mathbb{C}$, varies, are not bounded.

The image of an infinitely real differentiable and non constant function need not be unbounded

- For example: cos (x), sin(x) are infinitely real differentiable in $\mathbb{R}$, varies, are bounded.

2 .8.Power Series Expansion

If f is holomorphic (infinitely differentiable (analytic)) in region $\Omega \subseteq \mathbb{C}$ and $f^{(k)}(z_0) = 0$, $\forall$ k≥0 then $f(z) =0$

(by Taylor's representation) $\forall$ z in open disk centre at z_0 contained in D.

If f is infinitely real differentiable (real analytic) in an interval and $f^{(k)}(x_0)=0$, $\forall$ k≥0 then but Taylor's series does not exist in a neighbourhood of x_0 .

- For example: Not every infinite differentiable function has a power series

$f(x) = \begin{cases} e^{-1/x^2} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ is infinitely real differentiable at x =0 and $f^{(k)}(x_0)=0$ $\forall$ k≥0

$P^f(x) = \sum 0x^n = 0, and\ f(x) \neq P^f(x)\ for\ any\ x \neq 0\ (even\ though\ P^f(x)\ converges\ everywhere).$

2 .9.H (Ω) is integral domain w.r.t point wise addition & multiplication:

If $f(z)$, $g(z)$ are analytic in a region $\Omega \subseteq \mathbb{C}$ then $H(\Omega)$ is commutative ring with $1 \in H(\Omega)$ & without zero divisors

i.e. $f(z)g(z) = 0 \Rightarrow$ either $f(z) = 0$ or $g(z) = 0$ in Ω

- For example: Not true for real case, let I be an open connected subset of $\mathbb{R}$ and $f(x)$,$g(x)$ be real-differentiable functions on I such that $f(x)g(x) = 0$

$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$ $\quad g(x) = \begin{cases} 0 & \text{if } x \ge 0 \\ e^{-1/x^2} & \text{if } x < 0 \end{cases}$ Are such examples in C^∞ functions (but not real analytic.)

i.e. $f(x)g(x) = 0$ but neither $f(z) = 0$ nor $g(z) = 0$ in whole of $\mathbb{R}$.

2.10. Identity Theorem

States a complex-differentiable function on a domain vanishes on a non empty open set then it vanishes on entire domain,

- For example: Not true for real case, a real-differentiable function on $\mathbb{R}$ can take the value zero for some interval & then suddenly can take non-zero value.

$f(x) = \begin{cases} x^2, x \ge 0 \\ 0, x < 0 \end{cases}$ is real differentiable on $\mathbb{R}$ which is equal to zero on an open subset $(0,\infty)$ but still $f(x) \neq 0$ on whole $(-\infty,\infty)$.

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3.1. Applications of Real Analysis

1.Everything from ODEs and PDEs like Newton's second law, Wave equation ,Navier-Stokes equation, Maxwell equations ,Einstein field equations ,Schroedinger and Dirac equations etc, Taylor series, Fourier transforms, and in many functional decompositions.

2. It is essential for probability theory, which is the basics for all of statistics, queuing theory, operations research, and the mathematical finance, etc,

3. For business applications, real analysis is helpful in modeling consumer demand, churn, pricing, and advertising momentum. Some real analysis topics that arise in these applications are approximating functions through series expansions and inequalities, studying marginal effects through differentials, estimating long-term effects through limits and convergence, studying convexity and finding bounds, understanding optimization through successive approximation, etc.

3.2 Applications of Complex Analysis

Complex analysis is used in some major areas in engineering -

1.In signal processing, complex analysis and Fourier analysis go hand in hand in the analysis of signals, and this by itself has tones of applications, for instance, in communication systems (your broadband, wifi, satellite communication, image/video/audio compression, signal filtering/repair/reconstruction etc). Basically, if you search for applications of signal processing, screen projector s of pictures those are the applications that are indirectly the applications of complex analysis.

2.In control theory, complex analysis used specifically in the analysis of stability of systems and controller design. Here the word "system" is used generically, and does not necessarily refer to an electrical system. For instance, we can apply to understand stock market movement, chemical processes/reactions. Also, control theory is used heavily in robotics, and by extension etc.

3.Complex dynamics and the pictures of fractals produced by iterating analytic functions. Another important application of complex analysis is in string theory which studies conformal invariants in quantum field theory. And the capacitance or charge-holding capacity of condensers can be often be obtained by using conformal applying.

4.Many areas of maths and science routinely form real integrals that cannot be solved by the sorts of methods taught in real calculus courses but via complex integration we can replace intervals on the real line with curves in the complex plane to evaluate them.

IV CONCLUSION

In this paper, we first discussed in introduction that some fundamental properties like completeness, algebraically closured field, multiplication of complex no's, And there are several rigorous definition of limits ,differentiability, analytic properties such as power series expansion, geometric properties of analytic functions in one complex dimension (such as conformality) makes Complex analysis different from Real analysis. We embrace the differences rather than only finding ways to related it back to the Reals and are imperative to study carefully to teacher students in teaching field. This is paper presents an overview and the study of complex differentiability has much stronger consequences than usual (real) differentiability e.g. analytic functions are infinitely differentiable, whereas most real differentiable functions are not, The behavior of complex derivatives and differentiable functions is significantly different. And learned analytic functions exhibit some remarkable features.

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