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A REVIEW OF NEWTON'S METHOD FOR NONLINEAR TWO VARIABLE EQUATIONS

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ABSTRACT

In this paper solution of non linear system of equations is examined by Newton's method in one variable and also in two variables. Rate of convergence is examined and conclude that Newton's method converges rapidly when a good approximation is available. At last we found that a method with a higher rate of convergence may reach the solution of a system in less iteration in comparison to another method with a slower convergence.

Keywords: *Rate of convergence*

I. INTRODUCTION

Numerical methods are the study of methods in which we compute the numerical data. In this we find a general sequence of approximations with repeating the process again and again.

When we use the numerical methods for solving any problem .We want-

- The rate of convergence.
- Accuracy of the answer of the question.
- The completeness of the response.

In mathematics we study two types of equations-

- Linear system of equations
- Non- linear system of equations.

Linear System of Equations: Linear system of equation is represented by $\sum_{i=1}^{n} a_i x_i + b$ having n variables with degree 1.

Non Linear System of Equations : Non linear equation is also an algebraic equation which is not linear. These are of two types:

- Polynomial Equations
- Transcendental Equations

Solutions of Non Linear System of Equations

We obtain the solution of non linear system of equations by the following methods:

- Direct method
- Iterative method

Direct Method : When we find the roots of equation in a finite number of steps then it is called direct method like factorization, discriminate etc. Direct Method gives us an exact root of the equations.

Iterative Method : Iterative Methods based on successive approximations. In mathematics, an iterative method is a mathematical procedure which generates a sequence of improving approximate solutions. An iterative method is called convergent if any sequence converges for given initial approximations. An iterative method uses an initial guess to generate successive approximations to a solution.

In Numerical Methods, we can solve the equation with different methods:

- Bisection method
- Secant method & False position method
- Newton Raphson method

In this paper we will discuss Newton Raphson method in one variable as well as in two variables.

II. HISTORICAL BACKGROUND OF NEWTON RAPHSON METHOD

According to the Articles, N.Kollerstorm [11] or T.J.Ypma [12]. The following facts seem to be agreed upon among the experts:

Newton explained his method of approximation to the basic causes of numerical equations in a tract, In1600, Francois Vieta (1540–1603) had designed a perturbation technique for the solution of the scalar polynomial equations, which supplied one decimal place of the unknown solution per step via the explicit calculation of successive polynomials of the successive perturbations. In modern terms, the method converged linearly. It seems that this method had also been published in 1427 by the Persian astronomer and mathematician Al-Kashi (1380–1429). The Key to Arithmetic based on much earlier work by al-Biruni (973–1048); it is not clear to which extent this work was known in Europe. Around 1647, Vieta's method was simplified by the English mathematician Oughtred (1574–1660).

In 1664, Isaac Newton (1643–1727) got to know Vieta's method. Up to 1669 he had improved it by linearizing the successively arising polynomials.

Newton explained his method of approximation to the basic causes of numerical equations in a tract, *De analysi per aequations numeroterminaraun infinitas*. This is known as the first announcement of the principle of fluxions and binomial theorem.

In 1669, Newton placed it in the hands of his teacher, Isaac Barrow, then Barrow sent it to john Collins, he had a great desire for collecting and diffusing scientific information. John Collin was a member of Royal Society.

The earliest attempt of Newton's method of approximation became noticeable in Wallis Algebra, London, 1685 chapter 94.Wallis discusses Newton's method of solving the equation.

Some correspondents of Collins and friends of Newton knew about the tract but it was not printed until 1704 and 1711. Essentially, Newton gave same explanation of his method of approximation in his second tract, *The Methodus fluxionum et serierum infinitarum*. This planed for publication in 1671, but it was not printed until 1736.

As an example, he discussed the numerical solution of the cubic polynomial $f(x) = x^3 - 2x - 5 = 0.$

Newton first noted that the integer part of the root is 2. next; by means of $x = 2 + p$ he obtained the polynomial

$$p^3 + 6p^2 + 10p - 1 = 0.$$

Here he neglected terms higher than first order and thus put $p \approx 0.1$. He inserted $p = 0.1 + q$ and constructed the polynomial

$$q^3 + 6.3q^2 + 11.23q + 0.061 = 0$$

Again he neglected terms higher than linear and found $q \approx -0.0054$. Continuation of the process one more step led him to $r \approx 0.00004853$ and therefore to the third iterate

$$x_3 = x_0 + p + q + r = 2.09455147$$

Note that the relations $10p - 1 = 0$ and $11.23q + 0.061 = 0$ given above corresponds precisely to

$$p = x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

And to $$q = x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

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As the example shows, he had also observed that by keeping all decimal places of the corrections, the number of accurate places would double per each step.Quadratic convergence.

The extension of Newton-Raphson method to irrational and transcendental equations appear to have been made for the first time by Thomas Simpson with his Essays on Several Curious and Useful Subjects in Speculates and Mixed Mathematics', in London in 1740. In this he does not mention Newton and Raphson, he calls his procedure a 'New Method'. Thomas Simpson actually introduced derivatives in his book 'Essays on Several Curious and Useful Subjects in Speculates and Mixed Mathematics'. He described Newton's method as an iterative method for solving general non linear equations for one equation using fluxional calculus. Simpson also gives the generalization to system of two equations in two unknowns and shows that Newton method can also be used for solving optimization problems by setting the gradient to zero.

All 18^{th} and 19^{th} century writers discriminate between the methods of Newton and that of Raphson. Then the writers like Euler, Laplace, Lacroix and Legendre who explains the Newton-Raphson process. Finally, in different publications of writers distributed to Newton. Then popularity of Fourier's writings led to universal adoption of "Newton's method" for the Newton-Raphson process.

III. NEWTON METHOD

Newton's method is also called the Newton Raphson method. It is a root finding algorithm that uses the first few terms of the Taylor series of a function $f(x)$ in the vicinity of a suspected root. Newton's method is sometimes also known as Newton's iteration.

Newton Raphson Method In One Variable

Newton Raphson method is one of the fast iterative methods in Numerical Analysis. Newton Raphson method converges faster than false position method and secant method.

Let $f(x) = 0$ be the given equation.

Let x_k be an initial approximation to the root of the equation $f(x) = 0$.

Let Δx be an increment in x such that $x_k + \Delta x$ is an exact root where Δx is small.

$$\therefore f(x + \Delta x) = 0$$

Expanding $f(x + \Delta x)$ by Taylor series about the point x_k,

$$f(x_k + \Delta x) = f(x_k) + \Delta x f'(x_k) + \frac{1}{2!}(\Delta x)^2 f'(x_k) + ... = 0$$

Now Δx is small so that neglecting square and higher powers of Δx, where $f'(x_k) \neq 0.$

$$f(x_k) + \Delta x f'(x_k) = 0$$

$$\Delta x = -\frac{f(x_k)}{f'(x_k)}$$

Then the next approximation to the root is

$$x_{k+1} = x_k + \Delta x = x_k + \left(-\frac{f(x_k)}{f'(x_k)} \right)$$

$$= x_k - \frac{f(x_k)}{f'(x_k)}, \; k = 0,1,2,3.....$$

This formula is known as Newton Raphson Formula, its Rate of Convergence is 2.

Newton Raphson Method in Two Variables

Newton's method is one of the most popular numerical methods, and is even referred by Burden and Faires [13] as the most powerful method which is used for solving the equation $f(x) = 0$. Newton's method is that method which is related to a quadratic function. This approximate function is minimized exactly.

We can approximate function f at a given point x_k by a Taylor series

$$f(x) = f(x_k) + \nabla f(x_k)(x - x_k) + \frac{1}{2}(x - x_k)^t H^*(x_k)(x - x_k).$$

Where $H^*(x_k)$ is the Hessian matrix of a function f at a point x_k.

The minimization of a quadratic approximation f is that $\nabla f(x) = 0$.

It is assumed that the inverse of $H^*(x_k)$ exists, then the successor point $x_{k+!}$ is

$$x_{k+1} = x_k - \left[H^*(x_k) \right]^{-1} \nabla f(x_k)$$

And this equation is the recursive form of the points generated by Newton's method for the multi-dimensional non-linear optimization problem.

Assuming that $\nabla f(x^*) = 0$, that $H^*(x^*)$ is positive define at a local minimum x^* and f has a continuous second partial derivatives, $H^*(x)$ is positive definite at points which are near to x^* and then successor point x_{k+1} is well-defined.

IV. HESSIAN MATRIX

Hessian Matrix is square matrix of second order partial derivative of a scalar valued function, or a scalar field.

It was developed in 19th century by the German mathematician Ludwig Otto Hesse and later name after him. Hesse basically used the term "functional determinants".

Suppose $f: R^n \rightarrow R$ be a function is defined $f(x) \in R$ where $x \in R^n$

If all second order derivatives of function exist and continuous the domain of the function then the Hessian matrix H of f is a square $n \times n$ matrix

$$H = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \cdots\cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x^{22}} \cdots\cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} \cdots\cdots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix}, \quad H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

The determinant of the above matrix is also referred to as the Hessian.

The method was similar as we discuss for one variable. Here we take Taylor series expansion for two variables.

$$F(X) = f(X) + \sum_{j=1}^{n} \left(x_j - x_j' \right) \frac{\partial f}{\partial x_j} + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \left(x_j - x_j' \right) \left(x_i - x_i' \right) \frac{\partial^2 f}{\partial x_j \partial x_i},$$

V. RATE OF CONVERGENCE

Different methods in numerical analysis converge to the root at different rates. If we can begin with a good choice x_0 ,then Newton's method will converge to x^* rapidly.

The secant method is a little slower than Newton's method and Regula falsi method is slightly slower than that. However both are still much faster than the Bisection method.

The Regula falsi method, just like the bisection method, always works because it keeps the solution inside a definite interval.

VI. CONCLUSION:

We conclude that Numerical Methods are important methods of mathematics. They are powerful methods, not only solving nonlinear algebraic equations with one variable, but also the systems of nonlinear algebraic equations. Numerical methods are also used in solving for boundary value problems of nonlinear equations.

With regards to convergence, we can say that a numerical method with a higher rate of convergence may reach the solution of a system in less iteration in comparison to another method with a slower convergence. For example, Newton's method converge quadratic ally. The implication of this would be that given the exact same nonlinear system of equations denoted by F , Newton's method would arrive at the solution of $F = 0.$ in less iteration.

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