

α_T OPEN SETS IN TRI TOPOLOGICAL SPACE

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ABSTRACT

The main aim of this paper is to introduce new type of open sets namely α_T open sets in tri topological spaces along with their several properties and characterization. As application to α_T open sets we introduce α_T continuous functions and obtain some of their basic properties.

Keywords: α_T open sets, α_T continuous function.

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I. INTRODUCTION

In 1961 J. C. Kelly [II] introduced the concept of bitopological space. N. Levine [VI] introduced the idea of semi open sets and semi continuity and Mashhour et. al [VII] introduced the concept of pre open sets and pre continuity in a topological space. F.H. Khedr , S.M. Al-Areefi and T. Noiri [IV] generalize the notion of semi pre open sets to bitopological spaces and semi pre continuity in bitopological spaces. The concept of Pre-semi open sets (α open sets) was introduced by O. Njastad [VIII] in 1965. Tri topological space is a generalization of bitopological space . The study of tri-topological space was first initiated by Martin M. Kovar [V] .

S. Palaniammal [IX] study tri topological space. N.F. Hameed and Moh. Yahya Abid [I] gives the definition of 123 open set in tri topological spaces .U.D. Tapi , R. Sharma and B. Deole [X] introduce semi open set and pre open set in tri topological space. The purpose of the present paper is to introduce α_T open sets and α_T continuity and their fundamental properties in tri topological space. In this paper, tri open set is use in place of 123 open sets.

1. Preliminaries

Definition 1.1[VIII]: A subset A of a topological space (X, τ) is called pre-semi open set (α open sets) if $A \subseteq \text{int}(cl(\text{int} A))$. The complement of pre semi open set is called pre-semi closed set. The class of all pre-semi open sets of X is denoted by $PSO(X, \tau)$.

Definition 1.2[VIII]: Let (X, T_1) and (Y, T_2) are topological spaces. A function $f : X \rightarrow Y$ is called pre-semi continuous (α continuous) if the inverse image of each open set in Y is a pre-semi open set (α -open set) in X .

Definition 1.3[IX]: Let X be a nonempty set and T_1, T_2 and T_3 are three topologies on X .The set X together with three topologies is called a tri topological space and is denoted by (X, T_1, T_2, T_3)

Definition 1.4 [I]: A subset A of a topological space X is called 123 open set if $A \in T_1 \cup T_2 \cup T_3$ and complement of 123 open set is 123 closed set.

2. α_T open sets in Tri topological space

Definition 2.1: Let (X, T_1, T_2, T_3) be a tri topological space. A subset A of X is called α_T open in X , if $A \subseteq ps\text{int } pscl\text{ps int } A$. The complement of α_T open set is called α_T closed set. The collection of all α_T open sets of X is denoted by $PSO(X, T_1, T_2, T_3)$.

Example 2.2 Let $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a, b\}\}$, $T_2 = \{X, \phi, \{a\}\}$, $T_3 = \{X, \phi, \{b\}\}$

Tri open sets in tri topological spaces are union of all three topologies.

α_T open set of X is denoted by $PSO(X) = \{X, \phi, \{a\}, \{a, b\}, \{b\}\}$.

Let $A = \{a, b\}$;

$$\begin{aligned} T\text{int } Tc\text{I}T\text{int}\{a, b\} &= T\text{int } Tc\text{I}\{a, b\} \\ &= T\text{int } X \\ &= X \end{aligned}$$

$A = \{a, b\}$ is α_T open.

Definition 2.3: Let (X, T_1, T_2, T_3) be a tri topological space. Let $A \subset X$. An element $x \in A$ is called α_T interior point of A , if there exist a α_T open set U such that $x \in U \subset A$. The set of all α_T interior points of A is called the α_T interior of A and is denoted by $ps\text{int}(A)$.

Theorem 2.4: Let $A \subset X$ be a tri topological space. $ps\text{int}(A)$ is equal to the union of all α_T open sets contained in A .

Note 2.5: 1. $ps\text{int}(A) \subset A$.

2. $ps\text{int}(A)$ is α_T open sets.

Theorem 2.6: $ps\text{int}(A)$ is the largest α_T open sets contained in A .

Theorem 2.7: A is α_T open if and only if $A = ps\text{int}(A)$

Theorem 2.8: $ps\text{int}(A \cup B) \supset ps\text{int } A \cup ps\text{int } B$.

Definition 2.9: Let (X, T_1, T_2, T_3) be a tri topological space. Let $A \subset X$. The intersection of all α_T semi closed sets containing A is called a α_T closure of A and is denoted as $pscl(A)$.

Note 2.10 : Since intersection of α_T closed sets is α_T closed set, $pscl(A)$ is a α_T closed set.

Note 2.11: $pscl(A)$ is the smallest α_T closed set containing A .

Theorem 2.12: A is α_T closed set if and only if $A = pscl(A)$.

Theorem 2.13: Let A and B be subsets of (X, T_1, T_2, T_3) and $x \in X$

- a) A is α_T closed if and only if $A = pscl(A)$
- b) If $A \subset B$, then $pscl(A) \subset pscl(B)$.
- c) $x \in pscl(A)$ if and only if $A \cap U \neq \emptyset$ for every α_T open set U containing x .

Theorem 2.14: Let A be a subsets of (X, T_1, T_2, T_3) , if there exist an α_T open set U such that $A \subset U \subset pscl(A)$, then A is α_T open.

Theorem 2.15: In a tri topological space (X, T_1, T_2, T_3) , the union of any two α_T open sets is always an α_T open set.

Proof: Let A and B be any two α_T open sets in X .

$$\text{Now } A \cup B \subseteq pscl(ps\text{int}(A)) \cup pscl(ps\text{int}(B))$$

$$\Rightarrow A \cup B \subseteq pscl(\text{int}(A \cup B)). \text{ Hence } A \cup B \text{ is } \alpha_T \text{ open sets.}$$

Remark 2.16: The intersection of any two α_T open sets may not be a α_T open sets as show in the following example.

Example 2.17: Let $X = \{a, b, c, d\}$, $T_1 = \{X, \phi, \{a, d\}\}$, $T_2 = \{X, \phi, \{a, b, c\}\}$, $T_3 = \{X, \phi, \{b, c, d\}\}$

tri pre semi open set of X is denoted by $PSO(X) = \{X, \phi, \{a, b, d\}, \{a, b, c\}, \{b, c, d\}\}$.

Here $\{a, d\} \cap \{b, c, d\} = \{d\} \notin PSO(X)$.

Theorem 2.18: Let A and B be subsets of X such that $B \subseteq A \subseteq pscl(B)$. if B is α_T open set then A is also α_T open set.

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3. α_T Continuity in Tri topological space

Definition 3.1: A function f from a tri topological space (X, T_1, T_2, T_3) into another tri topological space

(Y, W_1, W_2, W_3) is called α_T continuous if $f^{-1}(V)$ is α_T open set in X for each tri open set V in Y .

Example 3.2: Let, $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a\}\}$, $T_2 = \{X, \phi, \{a, b\}\}$, $T_3 = \{X, \phi, \{b, c\}\}$

Open sets in tri topological spaces are union of all three topologies.

Then tri open sets of $X = \{X, \phi, \{a\}, \{a, b\}, \{b, c\}\}$.

$PSO(X)$ sets of $X = \{X, \phi, \{a\}, \{a, b\}, \{b, c\}\}$.

Let $Y = \{1, 2, 3\}$, $W_1 = \{Y, \phi, \{1, 3\}\}$, $W_2 = \{Y, \phi, \{2\}\}$, $W_3 = \{X, \phi, \{2\}, \{1, 2\}\}$

tri open sets of $Y = \{Y, \phi, \{2\}, \{1, 2\}, \{1, 3\}\}$.

$PSO(Y)$ sets of $Y = \{Y, \phi, \{2\}, \{1, 2\}, \{1, 3\}\}$.

Consider the function $f : X \rightarrow Y$ is defined as

$$f^{-1}\{2\} = \{a\}, f^{-1}\{1, 2\} = \{a, b\}, f^{-1}\{1, 3\} = \{b, c\}, f^{-1}(\phi) = \phi, f^{-1}(Y) = X.$$

Since the inverse image of each tri open set in Y under f is tri pre semi open set in X . Hence f is tri pre semi continuous function.

Theorem 3.3: Let $f : (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)$ be a α_T continuous open function. If A is an α_T open set of X , then $f(A)$ is α_T open in Y .

Proof : First, let A be α_T open set in X . There exist a tri open set U in X such that $A \subset U \subset pscl(A)$. Since f is α_T open function then $f(U)$ is tri open in Y . Since f is tri continuous function, we have $f(A) \subset f(U) \subset f(cl(A)) \subset cl(f(A))$. This shows that $f(A)$ is tri pre open in Y . Let A be tri pre open in X . There exist a tri pre open set U such that $U \subset A \subset (cl(U))$. Since f is tri continuous function, we have $f(U) \subset f(A) \subset f(cl(U)) \subset cl(f(U))$. By the proof of first part, $f(U)$ is tri pre open in Y . Therefore, $f(A)$ is tri pre open in Y .

Theorem 3.4: Let (X, T_1, T_2, T_3) and (Y, W_1, W_2, W_3) be two tri topological spaces. Then $f : X \rightarrow Y$ is pre semi continuous function if and only if $f^{-1}(V)$ is pre semi closed in X whenever V is tri closed in Y .

Theorem 3.5: Let (X, T_1, T_2, T_3) and (Y, W_1, W_2, W_3) be two tri topological spaces. A function $f : X \rightarrow Y$ is pre-semi continuous (α_T continuous) if and only if the inverse image of every α_T open set in Y is tri open in X .

Proof: (Necessary): Let $f : (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)$ be pre semi continuous function and U be any α_T open set in Y . Then $Y-U$ is α_T closed in Y . Since f is α_T continuous function, $f^{-1}(Y-U) = X - f^{-1}(U)$ is α_T closed in X and hence $f^{-1}(U)$ is α_T open in X .

(Sufficiency): Assume that $f^{-1}(V)$ is pre semi open in X for each tri open set V in Y . Let V be a closed set in Y . Then $Y-V$ is α_T open in Y . By assumption $f^{-1}(Y-V) = X - f^{-1}(V)$ is pre semi open in X which implies that $f^{-1}(V)$ is pre semi closed in (X, T_1, T_2, T_3) . Hence f is pre semi continuous function.

Theorem 3.6: Let $f : (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)$ be a α_T continuous open function. If V is an α_T open set of Y , then $f^{-1}(V)$ is α_T open in X .

Proof : First ,let V be α_T open set of Y .There exist an α_T set W in Y .such that $V \subset W \subset spcl(V)$.Since f is tri open set ,we have $f^{-1}(V) \subset f^{-1}(W) \subset f^{-1}(cl(V)) \subset cl(f^{-1}(V))$.since f is α_T continuous, $f^{-1}(W)$ is α_T open set in X .By theorem 4.18 , $f^{-1}(V)$ is α_T open set in X .The proof of the second part is shown by using the fact of first part.

Theorem 3.7: The following are equivalent for a function $f : (X, P_1, P_2, P_3) \rightarrow (Y, W_1, W_2, W_3)$:

- a) f is α_T continuous function ;
- b) the inverse image of each α_T closed set of Y is α_T closed in X ;
- c) For each $x \in X$ and each tri open set V in W containing $f(x)$,there exist an α_T open set U of X containing x such that $f(U) \subset V$;
- d) $pscl(f^{-1}(B)) \subset f^{-1}(pscl(B))$ for every subset B of Y .
- e) $f(pscl(A)) \subset pscl(f(A))$ for every subset A of X .

Theorem 3.8: If $f : (X, P_1, P_2, P_3) \rightarrow (Y, W_1, W_2, W_3)$ and $g : (Y, W_1, W_2, W_3) \rightarrow (Z, \eta_1, \eta_2, \eta_3)$ be two α_T continuous function then $gof : (X, P_1, P_2, P_3) \rightarrow (Z, \eta_1, \eta_2, \eta_3)$ may not be α_T continuous function .

Theorem 3.9: Let $f : (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)$ be bijective .Then the following conditions are equivalent:

- i) f is a α_T open continuous function.
- ii) f is α_T closed continuous function and
- iii) f^{-1} is α_T continuous function.

Proof:(i) \rightarrow (ii) Suppose B is a tri closed set in X .Then $X - B$ is an tri open set in X .Now by (i) $f(X - B)$ is a α_T open set in Y .Now since f^{-1} is bijective so $f(X - B) = Y - f(B)$.Hence $f(B)$ is a α_T closed set in Y .Therefore f is a α_T closed continuous function.

(ii) \rightarrow (iii) Let f is an α_T closed map and B be tri closed set of X .Since f^{-1} is bijective so $(f^{-1})^{-1}(B)$ which is an α_T closed set in Y . Hence f^{-1} is α_T continuous function.

(iii) \rightarrow (i) Let A be a tri open set in X .Since f^{-1} is a α_T continuous function so $(f^{-1})^{-1}(A) = f(A)$ is a α_T open set in Y . Hence f is α_T open continuous function.

Theorem 3.10: Let X and Y are two tri topological spaces. Then $f : (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)$ is α_T continuous function if one of the followings holds:

- i) $f^{-1}(psint(B)) \subseteq psint(f^{-1}(B))$, for every tri open set B in Y .

ii) $pscl(f^{-1}(B)) \subseteq f^{-1}(pscl(B))$, for every tri open set B in Y .

Proof: Let B be any tri open set in Y and if condition (i) is satisfied then $f^{-1}(psint(B)) \subseteq psint(f^{-1}(B))$.

We get $f^{-1}(B) \subseteq psint(f^{-1}(B))$. Therefore $f^{-1}(B)$ is a tri pre semi open set in X . Hence f is tri pre semi continuous function. Similarly we can prove (ii).

Theorem 3.11: A function $f : (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)$ is called α_T open continuous function if and only if $f(psint(A)) \subseteq psint(f(A))$, for every tri open set A in X .

Proof: Suppose that f is a α_T open continuous function.

since $psint(A) \subseteq A$ so $f(psint(A)) \subseteq f(A)$.

By hypothesis $f(psint(A))$ is an tri semi open set and $psint(f(A))$ is largest α_T open set contained in $f(A)$ so $f(psint(A)) \subseteq psint(f(A))$.

Conversely, suppose A is an tri open set in X . So $f(psint(A)) \subseteq psint(f(A))$.

Now since $A = psint(A)$ so $f(A) \subseteq psint(f(A))$. Therefore $f(A)$ is a α_T open set in Y and f is α_T open continuous function.

Theorem 3.12: A function $f : (X, T_1, T_2, T_3) \rightarrow (Y, W_1, W_2, W_3)$ is called α_T closed continuous function if and only if $pscl(f(A)) \subseteq f(pscl(A))$, for every tri closed set A in X .

Proof: Suppose that f is a α_T closed continuous function. since $A \subseteq pscl(A)$ so $f(A) \subseteq f(pscl(A))$. By hypothesis, $f(pscl(A))$ is a α_T closed set and $pscl(f(A))$ is smallest α_T closed set containing $f(A)$ so $pscl(f(A)) \subseteq f(pscl(A))$.

Conversely, suppose A is an tri closed set in X . So $pscl(f(A)) \subseteq f(pscl(A))$.

Since $A = pscl(A)$ so $pscl(f(A)) \subseteq f(A)$. Therefore $f(A)$ is a α_T closed set in Y and f is α_T closed continuous function.

Theorem 3.13: Let (X, T_1, T_2, T_3) and (Y, W_1, W_2, W_3) be two tri topological space. Then, $f : X \rightarrow Y$ is pre semi continuous function if and only if $f(pscl(A)) \subset pscl(f(A)) \forall A \subset X$.

Proof: Suppose $f : X \rightarrow Y$ is pre semi continuous function. Since $pscl[f(A)]$ is tri closed in Y . Then

by theorem (3.4) $f^{-1}[pscl(f(A))]$ is tri closed in X ,

$$pscl(f^{-1}(pscl(f(A)))) = f^{-1}(pscl(f(A))) \text{ --- (1)}$$

Now : $f(A) \subset pscl(f(A))$, $A \subset f^{-1}(f(A)) \subset f^{-1}(pscl(f(A)))$.

Then $pscl(A) \subset pscl(f^{-1}(pscl(f(A)))) = f^{-1}(pscl(f(A)))$ by (1).

Then $f(pscl(A)) \subset pscl(f(A))$.

Conversely, Let $f(pscl(A)) \subset pscl(f(A)) \forall A \subset X$.

Let F be α_T closed set in Y , so that $pscl(F) = F$. Now $f^{-1}(F) \subset X$, by hypothesis,

$$f(pscl(f^{-1}(F))) \subset pscl(f(f^{-1}(F))) \subset pscl(F) = F.$$

Therefore $pscl(f^{-1}(F)) \subset f^{-1}(F)$. But $f^{-1}(F) \subset pscl(f^{-1}(F))$ always.

Hence $pscl(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is α_T closed in X .

Hence by theorem (3.4) f is α_T continuous function.

Theorem 3.14 : Let (X, T_1, T_2, T_3) and (Y, W_1, W_2, W_3) be two tri topological spaces. Then, $f : X \rightarrow Y$ is α_T continuous function if and only if $pscl(f^{-1}(B)) \subset f^{-1}(pscl(B)) \forall B \subset Y$.

Proof: Suppose $f : X \rightarrow Y$ is α_T continuous. Since $pscl(B)$ is α_T closed in Y , then by theorem (3.4)

$$f^{-1}(pscl(B)) \text{ is } \alpha_T \text{ closed in } X \text{ and therefore, } pscl(f^{-1}(pscl(B))) = f^{-1}(pscl(B)) \dots \dots (2)$$

Now, $B \subset pscl(B)$, then $f^{-1}(B) \subset f^{-1}(pscl(B))$, then

$$pscl(f^{-1}(B)) \subset pscl(f^{-1}(pscl(B))) = f^{-1}(pscl(B)) \text{ by (2)}$$

Conversely: Let the condition hold and let F be any tri closed set in Y so that $pscl(F) = F$. By hypothesis, $pscl(f^{-1}(F)) \subset f^{-1}(pscl(F)) = f^{-1}(F)$. But $f^{-1}(F) \subset pscl(f^{-1}(F))$ always. Hence

$pscl(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is α_T closed in X . It follows from theorem (3.4) that f is α_T continuous function.

Theorem 3.15: Let (X, T_1, T_2, T_3) and (Y, W_1, W_2, W_3) be two tri topological spaces. Then, $f : X \rightarrow Y$ is α_T continuous function if and only if $f^{-1}(ps\text{int}(B)) \subset ps\text{int}(f^{-1}(B)) \forall B \subset Y$.

Proof: Let $f : X \rightarrow Y$ be a tri continuous. Since $\text{int}(B)$ is α_T open in Y , then by theorem (3.3)

$$f^{-1}(ps\text{int}(B)) \text{ is } \alpha_T \text{ open in } X \text{ and therefore, } ps\text{int}(f^{-1}(ps\text{int}(B))) = f^{-1}(ps\text{int}(B)) \dots \dots (3)$$

Now, $ps\text{int}(B) \subset B$, then $f^{-1}(ps\text{int}(B)) \subset f^{-1}(B)$, then $ps\text{int}(f^{-1}(ps\text{int}(B))) \subset ps\text{int}(f^{-1}(B))$ by (3)

Conversely: Let the condition hold and let G be any α_T open set in Y so that $ps\text{int}(G) = G$. By hypothesis, $f^{-1}(ps\text{int}(G)) \subset ps\text{int}(f^{-1}(G))$. Since $f^{-1}(ps\text{int}(G)) = f^{-1}(G)$ then

$$f^{-1}(G) \subset ps\text{int}(f^{-1}(G)) \text{ But } ps\text{int}(f^{-1}(G)) \subset f^{-1}(G) \text{ always and so}$$

$ps\text{int}(f^{-1}(G)) = f^{-1}(G)$. Therefore $f^{-1}(G)$ is α_T open in X . Consequently by theorem (3.3) f is α_T continuous function.

IV. CONCLUSION

We studied new form of pre semi open set in tri topological space. We also studied α_T continuous function in tri topological space. It is established that composition of any two tri pre semi open sets is again a tri pre semi open set in tri topological space. In tri pre semi continuity, inverse image of every tri open is tri pre semi open set.

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