

Impact of Commuting Probability Bounds in Quantum Systems

Parul Devi¹

¹Research Scholar, Glocal University, Saharanpur, U.P

Dr. Md. Heshamuddin²

²Assistant Professor, Glocal University, Saharanpur, U.P

ABSTRACT

Bounds for commuting probability play a crucial role in understanding the inherent probabilistic nature of quantum systems. This research paper delves into the concept of commuting probability in quantum systems and explores methods for calculating lower and upper bounds to quantify this fundamental property. Understanding commuting probability is crucial for characterizing the quantum dynamics of physical systems, and it has numerous applications in quantum information processing and quantum computing. By employing mathematical tools such as operator norms, commutation norms, Jensen's inequality, and spectral bounds, researchers can derive lower and upper bounds to quantify the likelihood of two observables commuting within specific quantum states. The investigation of these bounds not only enriches our understanding of quantum mechanics but also contributes to the development of efficient quantum algorithms and protocols.

Keywords: Bounds, Quantum, Operator, Commutative, Mechanics

I. INTRODUCTION

Bounds for commuting probability in quantum systems are a fascinating and fundamental topic that lies at the heart of quantum mechanics. Commuting probability, also known as the probability of two observables commuting, plays a pivotal role in understanding the dynamics and behavior of quantum systems. As quantum mechanics governs the behaviour of particles at the atomic and subatomic scales, it exhibits peculiar characteristics that often defy our classical intuitions. One such peculiarity is the concept of non-commutativity, wherein the order in which quantum observables are measured can impact the outcomes significantly. This departure from classical mechanics has profound implications for quantum information processing, quantum computing, and quantum cryptography.

In quantum mechanics, physical observables, such as position, momentum, spin, and energy, are represented by self-adjoint operators on a Hilbert space. When two observables do not affect each other's measurement outcomes, they are said to commute. Mathematically, two operators A and B commute if their commutator $[A, B] = AB - BA$ is equal to zero. This property, while seemingly

innocuous, gives rise to peculiar phenomena in quantum systems. The concept of commuting probability delves into quantifying the likelihood that two observables will commute given certain quantum states or circumstances.

The study of commuting probability is of paramount importance for several reasons. First and foremost, it sheds light on the fundamental principles that govern quantum mechanics. The fact that not all observables commute emphasizes the non-classical nature of quantum systems and challenges our classical intuitions about causality and measurement. Understanding when and why observables may commute or fail to do so enriches our grasp of the underlying mathematics and conceptual framework of quantum mechanics.

Furthermore, commuting probability has practical implications for quantum information processing and quantum computing. In quantum algorithms, the ability to commute observables can lead to significant speed-ups and efficiencies in solving certain computational problems. Conversely, the lack of commuting observables can introduce complexities in designing quantum algorithms and optimizing their performance. Therefore, characterizing the commuting probability for specific quantum states or quantum algorithms becomes essential for harnessing the full potential of quantum computation.

The exploration of bounds for commuting probability is a natural progression in the quest for a deeper understanding of quantum systems. These bounds provide valuable insights into the extreme scenarios of commuting probability, encompassing both the lower and upper limits of this probabilistic phenomenon. Deriving tight bounds allows researchers to ascertain the range within which commuting probability can vary, which is of particular importance when dealing with experimental uncertainties and imperfections in real-world quantum systems.

II. COMMUTING PROBABILITY IN QUANTUM SYSTEMS

Commuting probability in quantum systems is a fundamental concept that relates to the behavior of observables and their inherent probabilistic nature. In quantum mechanics, physical observables, such as position, momentum, energy, and spin, are represented by self-adjoint operators on a Hilbert space. The notion of commuting probability arises from the relationship between these operators and their commutativity.

The commuting probability, denoted as P_{comm} , quantifies the likelihood that two observables A and B will commute within a given quantum state ρ . Mathematically, P_{comm} is given by the expression $P_{\text{comm}} = \text{Tr}(\rho * \Pi_{\text{comm}})$, where Tr denotes the trace operation and Π_{comm} is the projector onto the subspace of states where the commutator $[A, B]$ is zero. The value of commuting probability lies within the range $[0, 1]$, where $P_{\text{comm}} = 1$ indicates that the observables A and B always commute, and $P_{\text{comm}} = 0$ signifies that they never commute. For a pure quantum state, P_{comm} can reach its maximum value of 1 if the observables A and B have a common set of eigenvectors. Conversely, for

mixed states, the commuting probability can take intermediate values between 0 and 1. Commuting probability has significant implications in quantum mechanics and various applications in quantum information processing and quantum computing. Deriving bounds for commuting probability is a valuable endeavor to understand the extremes of commutativity in quantum systems. Employing mathematical tools such as operator norms, commutation norms, Jensen's inequality, and spectral bounds allows researchers to quantify the likelihood of commuting observables within specific quantum states. These bounds not only enrich our understanding of quantum dynamics but also contribute to the development of efficient quantum algorithms and protocols.

III. SOME LOWER BOUNDS

In this section, we derive some lower bounds for $\text{Pr}(R)$ and show that our bounds are better than some existing lower bounds for $\text{Pr}(R)$. We begin with the following theorem.

Theorem: Let R be a finite non-commutative ring. If p is the smallest prime dividing $|R|$ then

$$\text{Pr}(R) \geq \frac{|Z(R)|}{|R|} + \frac{p(|R| - |Z(R)|)}{|R|^2}$$

with equality if and only if $|C_R(r)| = p$ for all $r \notin Z(R)$.

Proof. By (above eq.), we have

$$|R|^2 \text{Pr}(R) = |R||Z(R)| + \sum_{R/Z(R)} |C_R(r)|$$

with equality if and only if $|C_R(r)| = p$ for all $r \notin Z(R)$. Hence, the result follows from (above eq.).

The following two results give some improvements of Result.

Theorem: Let R be a finite ring. Then

$$\text{Pr}(R) \geq \frac{1}{|K(R, R)|} \left(1 + \frac{|K(R, R)| - 1}{|R:Z(R)|} \right)$$

with equality if and only if $|K(R, R)| = |[r, R]|$ for all $r \in R/Z(R)$. In particular, if R is non-commutative then $\text{Pr}(R) > \frac{1}{|K(R, R)|}$.

Proof. By (above eq.), we have

$$\text{Pr}(R) = \frac{|Z(R)|}{|R|} + \frac{1}{|R|} \sum_{R/Z(R)} \frac{1}{|R:C_R(r)|}$$

Since $|K(R, R)| \geq |[r, R]| = |R : C_R(r)|$ for all $R/Z(R)$, we have

$$\text{Pr}(R) \geq \frac{|Z(R)|}{|R|} + \frac{1}{|R|} \sum_{R/Z(R)} \frac{1}{|K(R, R)|} = \frac{|Z(R)|}{|R|} + \frac{|R| - |Z(R)|}{|R||K(R, R)|}$$

IV. SOME UPPER BOUNDS

Here we compute two upper bounds on $\text{Pr}(R)$. Specifically, we will demonstrate that our constraints for Pr are more conservative than certain current bounds (R).

Theorem: Let R be a finite non-commutative ring. If p is the smallest prime dividing $|R|$ then

$$\text{Pr}(R) \leq \frac{(p-1)|Z(R)| + |R|}{p|R|}$$

with equality if and only if $|R : \text{CR}(r)| = p$ for all $r \notin Z(R)$.

Proof. By (above eq.), we have

$$|R|^2 \text{Pr}(R) = |R||Z(R)| + \sum_{R/Z(R)} |C_R(r)|$$

If $r \notin Z(R)$ then

$$|C_R(r)| \leq \frac{|R|}{p}$$

Therefore

$$\sum_{R/Z(R)} |C_R(r)| \leq \frac{|R|(|R| - |Z(R)|)}{p}$$

with equality if and only if $|R : \text{CR}(r)| = p$ for all $r \notin Z(R)$. Hence, the result follows from (above eq.).

If R is a non-commutative ring and p the smallest prime dividing $|R|$ then $|R : Z(R)| \geq p^2$. Therefore

$$\frac{(p-1)|Z(R)| + |R|}{p|R|} \leq \frac{p^2 + p + 1}{p^3}$$

Thus the upper bound obtained in Theorem is better than Result.

V. MATHEMATICAL FRAMEWORK FOR BOUNDS CALCULATION

Commuting Operators

In quantum mechanics, observables are represented by self-adjoint operators on a Hilbert space. Two operators A and B are said to commute if their commutator $[A, B] = AB - BA$ is equal to zero. Mathematically, this can be written as $[A, B] = 0$.

Operator Norms

The operator norm $\|A\|$ of an operator A on a Hilbert space is a measure of its size and can be defined in various ways, such as the operator norm induced by the inner product or the Schatten norms. These norms are essential in calculating bounds for commuting probability as they provide a way to quantify the "distance" between two operators.

Commutation Norm

The commutation norm of two operators A and B is defined as $\|[A, B]\|$, i.e., the norm of their commutator. The commutation norm measures how much two operators fail to commute and is a crucial quantity in deriving bounds for commuting probability.

Jensen's Inequality

Jensen's inequality is a fundamental concept in probability theory and functional analysis. It states that for a convex function $f(x)$, if X is a random variable, then $E[f(X)] \geq f(E[X])$, where $E[\cdot]$ denotes the expected value. This inequality is employed in deriving both lower and upper bounds for commuting probability.

Spectral Bounds

The spectral bounds of an operator A are determined by its eigenvalues. The minimum and maximum eigenvalues of A , denoted by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively, play a significant role in calculating bounds for commuting probability.

VI. CONCLUSION

The study of bounds for commuting probability in quantum systems represents a pivotal and dynamic field within quantum mechanics. Through a diverse array of mathematical tools and approaches, researchers have uncovered invaluable insights into the probabilistic nature of quantum observables and their propensity to commute or not. The pursuit of bounds for commuting probability not only enriches our fundamental knowledge of quantum mechanics but also holds the key to unlocking the transformative power of quantum technologies. By advancing our understanding of commuting probability and its bounds, researchers are paving the way for a future of unprecedented technological advancements and groundbreaking discoveries in the quantum realm.

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