International Journal of Innovative Research in Science and Engineering
Volume 11, Issue 07, 2025
www.ijirse.com
ISSN: 2454-9665

FIXED-POINT THEOREMS FOR FUZZY MAPPINGS ON PARTIALLY ORDERED METRIC SPACES

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Abstract

This paper studies fixed-point and common fixed-point results for monotone mappings in fuzzy partially ordered metric spaces. The fuzzy metric framework represents nearness by a degree depending on a positive parameter, while the partial order supports the construction of monotone iterative sequences. Under completeness, monotonicity and suitable fuzzy contractive assumptions, existence and uniqueness of fixed points are established. A common fixed-point result for compatible mappings is also presented. An asymptotically fuzzy contractive setting and an Ishikawa-type iterative approximation are briefly considered. The results illustrate how fuzzy nearness and order structure can be combined to extend classical fixed-point methods.

Keywords: Fuzzy metric space; partially ordered set; fixed point; common fixed point; monotone mapping; fuzzy contraction; compatible mappings; Ishikawa iteration.

1. Introduction

Fixed-point theory is an important tool in nonlinear analysis, functional analysis, differential equations, optimization and mathematical modelling. Classical metric spaces may be restrictive when nearness is gradual or uncertain. Fuzzy metric spaces provide a natural framework in which the degree of nearness between two points is described as a value in [0,1]. The incorporation of a partial order further assists in constructing monotone iterative sequences and in controlling their limiting behavior. Motivated by these ideas, this paper presents a concise

International Journal of Innovative Research in Science and Engineering
Volume 11, Issue 07, 2025
www.ijirse.com
ISSN: 2454-9665

framework for fixed points and common fixed points of fuzzy mappings on partially ordered metric spaces, together with an asymptotic extension and an iterative approximation result. [1-6]

2. Preliminaries

A fuzzy metric space is a triple $(X, M, *)$ where X is non-empty, $*$ is a continuous t-norm, and $M:X\times X\times(0, \infty)\to[0, 1]$ satisfies, for $x, y, z\in X$ and $s, t>0$:

(i) $M(x, y, t)>0$;

(ii) $M(x, y, t) = 1$ if and only if $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, z, t+s) \geq M(x, y, t)*M(y, z, s)$; and

(v) $t\mapsto M(x, y, t)$ is continuous. The value $M(x, y, t)$ represents the degree of nearness of x and y at parameter t.

If $\preceq$ is a partial order on X, then $(X, M, *, \preceq)$ is called a fuzzy partially ordered metric space. A mapping $T:X\to X$ is monotone non-decreasing if $x\preceq y$ implies $T(x)\preceq T(y)$. We call T a fuzzy contraction when there is $k\in(0,1)$ such that $M(Tx,Ty,t)\geq M(x,y,kt)$ for comparable x,y and $t>0$. Two self-mappings T and S are compatible if, whenever $Tx_n\to z$ and $Sx_n\to z$, $M(TSx_n, STx_n, t)\to 1$. A point x^* is a common fixed point if $T(x^*)=S(x^*)=x^*$.

A basic example is $X=\mathbb{R}$ with the product t-norm $a*b=ab$ and $M(x, y, t)=t/(t+|x-y|)$. On $[0,1]$ with the usual order, this gives a fuzzy partially ordered metric space. For $T(x)=x/2$, the mapping is monotone non-decreasing and satisfies a fuzzy contraction condition with $k=1/2$.

3. Main Fixed-Point Result

Theorem 3.1 (Existence of a Fixed Point).

Let $(X,M,*,\preceq)$ be a complete fuzzy partially ordered metric space [2]. Suppose $T:X\to X$ is monotone non-decreasing and there exists $k\in(0,1)$ such that, for all comparable $x,y\in X$ and $t>0$,

$$M(Tx,Ty,t) \geq \min\{M(x,y,kt), M(x,Tx,kt), M(x,Ty,kt)\}.$$

Assume also that there is $x_0\in X$ with $x_0\preceq T(x_0)$. Then T has a fixed point in X.

Proof.

Choose $x_0 \in X$ with $x_0 \preceq T(x_0)$, and define an iterative sequence by

$$x_{n+1} = T(x_n), \quad n \geq 0.$$

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Step 1. Monotonicity of the sequence.

Since T is monotone non-decreasing and $x_0 \preceq T(x_0)$, we obtain $x_0 \preceq x_1 = T(x_0)$. Applying monotonicity again gives $x_1 = T(x_0) \preceq T(x_1) = x_2$. Continuing inductively, we obtain

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

Hence $\{x_n\}$ is a non-decreasing sequence in $(X, \preceq)$.

Step 2. Contractive estimate.

Using the fuzzy contractive condition with $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Tx_{n-1}, Tx_n, t) \\ &\geq \min\{M(x_{n-1}, x_n, kt), M(x_n, Tx_n, kt), M(x_{n-1}, Tx_{n-1}, kt)\} \\ &= \min\{M(x_{n-1}, x_n, kt), M(x_{n-1}, x_n, kt), M(x_{n-1}, x_n, kt)\} \\ &= \min\{M(x_{n-1}, x_n, kt), M(x_{n-1}, x_n, kt), M(x_{n-1}, x_n, kt)\} \\ &= M(x_{n-1}, x_n, kt) \end{aligned}$$

Iterating the same estimate yields

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t).$$

Step 3. Cauchy property.

Let $m > n$. By the triangle inequality property of the fuzzy metric,

$$M(x_n, x_m, t) \geq M(x_n, x_{n+1}, t/m) * M(x_{n+1}, x_{n+2}, t/m) * \cdots * M(x_{m-1}, x_m, t/m).$$

Each factor on the right tends to 1 as $n \to \infty$; therefore $M(x_n, x_m, t) \to 1$ as $n, m \to \infty$. Thus $\{x_n\}$ is a fuzzy Cauchy sequence.

Step 4. Completeness.

Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Because the sequence is increasing and the order is compatible with the limit process, we also have $x_n \preceq x^*$ for all n.

Step 5. Verification of the fixed-point property.

From $x_{n+1} = T(x_n)$ and the contractive inequality, we obtain

$$M(Tx_n, Tx^*, t) \geq \min\{M(x_n, x^*, kt), M(x_n, T x_n, kt), M(x_n, Tx^*, kt)\} \quad t > 0.$$

Letting $n \to \infty$ gives $M(Tx^*, x^*, t) = 1$ for every $t > 0$, and hence $Tx^* = x^*$. Therefore x^* is a fixed point of T. This completes the proof.

Corollary 3.2 (Uniqueness).

Under the assumptions of Theorem 3.1, suppose in addition that, for all comparable $x \neq y$ and every $t > 0$. Then the fixed point of T is unique.

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Proof.

Let p and q be comparable fixed points with $p \preceq q$. Then

$M(p, q, t) = M(Tp, Tq, t)$

$\geq \min\{ M(p, q, kt), M(p, Tp, kt), M(p, Tq, kt)\}$

$> M(p, q, t)$

which is impossible. Hence $p = q$.

Theorem 3.3 Common Fixed Point for Two Mappings [4].

Let $(X, M, *, \preceq)$ be a complete fuzzy partially ordered metric space and let $T, S : X \to X$ satisfy:

1. T and S are monotone non-decreasing;
2. there exists a non-decreasing function $\varphi : [0, 1] \to [0, 1]$ such that $\varphi(s) > s$ for $0 < s < 1$ and $M(Tx, Sy, t) \geq \varphi\min\{M(x, y, t), M(x, Tx, t), M(x, Sy, kt)\}$ for all comparable $x, y \in X$ and every $t > 0$;
3. there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$ and $x_0 \preceq S(x_0)$;
4. the pair (T, S) is compatible.

Then T and S have a common fixed point in X.

Proof.

Choose $x_0 \in X$ such that $x_0 \preceq T(x_0)$ and $x_0 \preceq S(x_0)$. Define the sequence $\{x_n\}$ by

$$x_{2n+1} = T(x_{2n}), \quad x_{2n+2} = S(x_{2n+1}), \quad n \geq 0.$$

Step 1. Monotonicity.

Because T and S are monotone non-decreasing, we obtain

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

Thus $\{x_n\}$ is non-decreasing.

Step 2. Successive contractive estimates.

For every n, the contractive assumption gives

$M(x_{n+1}, x_{n+2}, t) = M(Tx_n, Sx_{n+1}, t)$

$\geq \varphi\min\{M(x_n, x_{n+1}, t), M(x_n, Tx_n, t), M(x_n, Sx_{n+1}, kt)\}$

Since φ is non-decreasing and $\varphi(s) > s$ on (0, 1), repeated application forces $M(x_n, x_{n+1}, t) \to 1$ for every $t > 0$. By Lemma 2.3.1, $\{x_n\}$ is a Cauchy sequence.

Step 3. Passage to the limit.

Completeness yields a point $x^* \in X$ such that $x_n \to x^*$. Since the sequence is increasing, we have $x_n \preceq x^*$ for all n. Compatibility of T and S implies

$$M(TSx_n, STx_n, t) \to 1, \quad t > 0,$$

and by taking limits we obtain $T(x^*) = S(x^*)$. Using the iterative definition and the convergence of both subsequences, we further conclude that $T(x^*) = x^*$ and $S(x^*) = x^*$. Hence x^* is a common fixed point of T and S.

Corollary 3.4 Weak Compatibility Version [5].

If compatibility in Theorem 3.3 is replaced by weak compatibility, and the iterative limit x^* is a coincidence point of T and S, then x^* is a common fixed point of T and S.

4. Asymptotically Fuzzy Contractive Mappings

An asymptotic version is obtained when there exists a sequence $\lambda_n>1$ with $\lambda_n\to\infty$ such that, for comparable x, y,

$$M(T^nx,T^ny,t) \geq \min\{M(x,y,\lambda_n t), M(x,T^nx,\lambda_n t), M(x,T^ny,\lambda_n t)\}.$$

If T is monotone and continuous, $x_0\preceq T(x_0)$, and the orbit $\{T^nx_0\}$ is Cauchy, then completeness gives $x_n\to x^*$ and continuity gives $T(x^*)=x^*$. If fixed points are comparable, letting $n\to\infty$ in the asymptotic estimate yields uniqueness.

5. Ishikawa-Type Approximation [6]

For a unique fixed point p, an Ishikawa-type iteration can be used for approximation. Let $\{a_n\},\{b_n\}\subset[0,1]$, with $\Sigma a_n=\infty$, and define

$$y_n=(1-b_n)x_n\oplus b_nT(x_n), \quad x_{n+1}=(1-a_n)x_n\oplus a_nT(y_n).$$

Under the hypotheses ensuring uniqueness and the appropriate fuzzy convex structure, the iterative sequence converges to p. When $b_n=0$, the scheme reduces to Mann iteration. For the model example $T(x)=x/2$ on $[0,1]$, the fixed point is $p=0$.

6. Examples

Example 6.1. Let $X=[0,1]$, $M(x,y,t)=t/(t+|x-y|)$, and $T(x)=x/2$. Then T is monotone non-decreasing, satisfies the fuzzy contraction condition with $k=1/2$, and has the unique fixed point 0.

Example 6.2. Let $T(x)=x/2$ and $S(x)=x/4$ on $[0,1]$. Both are monotone and weakly compatible, and their unique common fixed point is 0.

Example 6.3. On $X=[0,1]\cup[2,3]$ with the same fuzzy metric, define $T(x)=x/2$ on $[0,1]$ and $T(x)=2+(x-2)/2$ on $[2,3]$. Then 0 and 2 are fixed points, illustrating that separate ordered components can produce multiple fixed points when global comparability is absent.

7. Conclusion

A concise fixed-point framework for fuzzy mappings on partially ordered metric spaces has been presented. Completeness, monotonicity and fuzzy contractive conditions provide existence and uniqueness of fixed points, while compatibility supports common fixed-point results for pairs of mappings. An asymptotically fuzzy contractive setting and an Ishikawa-type iteration extend the framework toward generalized and computational settings. The examples demonstrate the interaction between fuzzy nearness and order structure.

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