International Journal of Innovative Research in Science and Engineering
Vol. No.3, Issue 04, April 2017
www.ijirse.com
ISSN (O) 2454-9665
ISSN (P) 2454-0663

AN INTRODUCTION TO CURVATURE AND RIEMANNIAN CURVATURES AND SOME DISCRETE GENERALIZATIONS

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ABSTRACT

An introduction of curvature and Riemannian curvature is provided to some entities used in Riemannian geometry such as sectional curvature, Ricci curvature, parallel transport, Bianchi identities . Some of the approaches is used to define equivalents of curvature in non-smooth or discrete spaces is tried to explain.

In the first part of this text tried to explain intuitively some of the entities used in Riemannian geometry such as sectional curvature, Ricci curvature, parallel transport, Bianchi identities . Usual entities of Riemannian geometry: sectional curvature, Ricci curvature, parallel transport, Bianchi identities for each of those we tried to interpret the formal definition. In the spirit of coarse geometry in which we investigate large scale properties of metric spaces instead of small-scale properties we consider the problem of defining analogues of curvature for non-smooth or discrete objects. One of the aims being to be impervious to small perturbations of the underlying space. Various other fields of mathematics, such as group theory or optimal transport motivated us for these non-smooth or discrete extensions. Metric measure spaces theory emerges from this viewpoint[Gro99]..

I. CURVATURE IN HIGHER DIMENSIONS

Suppose M is an n-dimensional manifold equipped with a Riemannian metric g. As with surfaces, the basic geometric invariant is curvature, but in higher dimensions curvature becomes a much more complicated quantity because a manifold may curve in so many directions.

Riemannian manifolds are not presented to us as embedded submanifolds of Euclidean space. Hence we must not consider the idea of cutting out curves by intersecting our manifold with planes, as we did when defining the princi-pal curvatures of a surface in R^3. Geodesics are the curves that are the shortest paths between nearby points—are ready-made tools for this and many other purposes in Riemannian geometry. Examples are straight lines in Euclidean space and great circles on a sphere.

The most fundamental fact about geodesics, is that given any point $p \in M$ and any vector V tangent to M at p, there is a unique geodesic starting at p with initial tangent vector V .

Here is a brief recipe for computing some curvatures at a point $p \in M$:

1. Pick a 2-dimensional subspace Π of the tangent space to M at p.

Look at all the geodesics through p whose initial tangent vectors lie in the selected plane Π. It turns out that near p these sweep out a certain 2-dimensional submanifold $S\Pi$ of M, which inherits a Riemannian metric from M.

Compute the Gaussian curvature of SΠ at p, which the Theorema Egregium tells us can be computed from its Riemannian metric. This gives a number, denoted K(Π), called the sectional curvature of M at p associated with the plane Π.

Thus the "curvature" of M at p has to be interpreted as a map

K : {2-planes in TpM} → R.

II. RIEMANNIAN CURVATURES

Riemannian manifolds: A smooth surface embedded in three-dimensional Euclidean space is called a Riemannian manifold. More gener-ally, any smooth manifold consists of a set $X \subset R^n$ for some integer n, such that, at any point $x \in X$, there exists an N -dimensional affine subspace of R^n which corresponds with X at first order around x. This subspace is the **tangent space** T_xX at $x \in X$, and N is the **dimension** of X.

If t →c(t) is a smooth curve inside X, then its derivative dc(t)/dt is a vector tangent to X at c(t).

A **Riemannian** manifold is a manifold equipped with a **Riemannian metric**, that is, for each $x \in X$, a definite positive quadratic form defined on T_xX. For instance, if X is included in R^n, such a quadratic form might be the restriction to T_xX of the canonical Euclidean structure on R^n. We will assume that the quadratic form depends smoothly on $x \in X$.

The norm of this vector is defined by a quadratic form which is applied to any tangent vector. Integration of which can then define the length of a curve in X. The infimum of the lengths of all curves between these two points is then defined as the distance (inside X) between two points of X. This turns X into a metric space.

For this metric we will always assume that X is connected and complete.In X consider a curve γ in way that, for any two close points on γ, the distance in X between those two points is obtained by travelling along γ is called a geodesic. Such curves always exist locally. Given a tangent vector $x \in X$ and a tangent vector $v \in T_xX$ at x, there exists exactly one geodesic starting at x with its initial velocity is v and having constant speed. This will be the **geodesic starting along** v. After following the geodesic starting along v for a unit time the **endpoint** of v will be obtained and is denoted $\exp_x v$.

Parallel transport: We take two very close points x and y in a Riemannian manifold. In order to compare a tangent vector at x and a tangent vector at y, even though belong to different vector spaces can be achieved via parallel transport.

Considering w_x as a tangent vector at x; we will search for a tangent vector w_y at y which would be "the same" as w_x. As we know x and y are very close, we may assume the endpoint of a small tangent vector v at x is y. To make it simple, we will assume that the norm of w_x is very small and that w_x is orthogonal to v. Hence, there exists a particular tangent vector w_y at y which is the one whose endpoint is closest to the endpoint of w_x. Given the restriction that w_y be orthogonal to v (if the orthogonality condition is not there, then there is a tangent vector at y whose endpoint is exactly the endpoint of w_x, but this vector "turns back towards x"). The vector w_y is the best candidate to be "the same" as w_x, translated to y .

International Journal of Innovative Research in Science and Engineering
Vol. No.3, Issue 04, April 2017
www.ijirse.com
ISSN (O) 2454-9665
ISSN (P) 2454-0663

The operation mapping w_x to w_y is called Parallel transport . (Preciesly, by our have assumption that w_x is very small, we will consider the linear component of the map $w_x \to w_y$ for small w_x and then extend by linearity to a map between the whole spaces $T_x X$ and $T_y X$. Where w_x is orthogonal to v is assumed in the definition. The parallel transport of v from x to y is now defined as the velocity at y of the geodesic starting along v.

Generally, along any smooth curve starting at x parallel transport of a vector w can be defined by decomposing the curve into small intervals and by performing along these subintervals successive parallel transports.

Sectional curvature and Ricci curvature:As we are defining various curvatures. The first one we consider is sectional curvature.

Considering again x be a point in X, v a small tangent vector at x were y the endpoint of v having w_x a small tangent vector at x and the parallel transport of w_x from x to y along v is w_y. If, in lieu of a Riemannian manifold, we were working in ordinary Euclidean space, the endpoints of w_x and w_y are x' and y' would be a rectangle with x and y. But while working in manifold, generally these four points would not be a rectangle any more.

The two geodesics starting along w_x and w_y may diverge from or converge towards each other due to curvature. Due to this , on a sphere withpositive curvature, two meridians starting at two points on the equator have parallel initial velocities, yet they converge at the North and South poles. As we konow the initial velocities w_x and w_y are parallel to each other, this effect is at second order in the distance along the geodesics .

Let us consider the points lying at small distance ε from x and y on the geodesics starting along w_x and w_y , respectively., The distance between those two points would be |v| in a Euclidean setting ,the same as the distance between x and y. The disagreement from this Euclidean case is used as a definition of a curvature.

DEFINITION 1.1 (Sectional curvature). Let (X, d) be a Riemannian manifold. Considering v and w_x be two unit-length tangent vectors at some point $x \in X$. Let $\varepsilon, \delta > 0$. Let y be the endpoint of δv and let w_y be obtained by parallel transport of w_x from x to y. Then

$$\overline{\varepsilon^2}\, d(\exp_x \varepsilon w_x, \exp_y \varepsilon w_y) = \delta\ 1 - 2\ K(v, w) + O(\varepsilon^3 + \varepsilon^2 \delta)$$

when $(\varepsilon, \delta) \to 0$. This defines a quantity K(v, w), which is the sectional curvature at x in the directions (v, w).
Instead **Ricci curvature** depends only on one tangent vector v; which is obtained by averaging K(v, w) over all the directions w .

DEFINITION 1.2 (Ricci curvature). In an N -dimensional Rie-mannian manifold, let x be a point. Considering v as a unit tangent vector at x. The quantity Ric(v) which is defined as Ricci curvature along v is the N times the average of K(v, w), where the average is taken over w running over the unit sphere in the tangent space $T_x X$.

The conventional definition of Ricci curvature provides the scaling factor N as the trace of a linear map, which results in a sum over a basis rather than an average over the unit sphere. In fact the quadratic form in v is a nothing but Ric(v) and is frequently denoted by Ric(v, v) and the Ricci tensor is the corresponding bilinear form $Ric(v, v')$.

The Riemann curvature tensor: The three tangent vectors u, v, w at a given point x is the basis of the riemann curvature tensor and R(v, w)u is its output is another tangent vector at x. Prior to we define this tensor, we need to speak more about parallel transport.

At point x let u and v be two tangent vectors , we may parallel-transport w along v; this yields a tangent vector w' at the endpoint of v. Similarly, we may also parallel-transport v along w, getting a tangent vector v' at the endpoint of w. In Euclidean space, the vectors v, w, v', w' constitute a parallelogram; in particular, the endpoints of v′ and of w′ coincide. But we could expect that, in a general Rie-mannian manifold, the endpoints of v′ and w′ may not coincide anymore. Actually for small v and w, the endpoints of v′ and w′ do coincide at the first order at which this is non-trivially true (actually, up to o(|v| |w|)). torsion-freeness is the property of parallel transport is called and expresses the fact that "parallelograms close up".

Actually this property was used perfectly before when we defined curvature, we had to consider a phenomenon of order $O(|v|\,|w|^2)$.

Let us now consider three vectors u, v, w instead of two. Let us keep the par-allelogram built from u and v, and let us parallel-transport u along the path vw, or along the path wv. We obtain two different results and their difference is the **Riemann curvature** R(v, w)u, which is again a tangent vector .

This proves that, in contrast to parallelograms, cubes don't close up in Rie-mannian geometry.

On further investigation of the given three directions u, v, w for the sides of a cube whose base corner is x. We get several ways to build the farthest corner of the cube. We observe that the paths vwu and wvu do not end up at the same point, and that the difference is R(v, w)u. (When we mention "the path vwu", it implies the tangent vector v to its endpoint and the parallel-transport w along this path .The obtained vector to its endpoint, parallel-transport u along the whole path, and follow the obtained vector to its endpoint.)

By considering given u, v and w, there are six possible ways to define the farthest corner by six possible paths to the farthest corner. The three distinct endpoints is defined by these six points.We know , torsion-freeness guarantees that vw = wv, but at the endpoint of u torsion-freeness also guarantees that uvw = uwv. The latter equality just implies that "lateral faces of the cube close up".

A triangle is formed by three distinct endpoints. By definition,we observed that one of the sides of this triangle is R(v, w)u and symmetry gives us the other sides as R(w, u)v and R(u, v)w. As the three sides of any triangle add up to 0 we get

$$R(u, v)w + R(v, w)u + R(w, u)v = 0$$

which is the **first Bianchi identity** and was discovered by Ricci.

Riemannian volume measure: Another notion of Riemannian geometry is that of Riemannian volume measure which is a measure on an N - dimensional Riemannian manifolds which gives volume ε^N to a small cube made of N tangent vectors at the same point, each of length ε and orthogonal to each other.

III. DISCRETE CURVATURES

Let us now consider more general spaces than smooth manifolds and apply generalizations of these notions to them . This is stimulated by the thought of "synthetic" ge-ometry: a geometry that focuses on "large-scale" as an option of "small-scale" properties, and uses mainly metric comparisons, avoiding local properties such as differentia-bility, with the goal of being invariant under small trepidations of the space underlying. We refer for instance to Gromov's influential book [Gro99].

Considering two types of spaces. Continuous spaces in which the distance between points is always realized by the length of a continuous curve are **Geodesic spaces** .Explicit examples are objects obtained as limits of smooth manifolds ,or,piecewise smooth objects, which can be notion of as manifolds with singularities.

Discrete spaces : Distinctive examples are discrete groups equipped with a Cayley graph metric, or spaces of configurations of discrete statis-tical physics systems.

ISSN (O) 2454-9665
ISSN (P) 2454-0663

The generalizations of sectional curvature which are now implicit especially in negative curvature, with deep connections for illustration with group theory. Our centre of attention is on generalizations of Ricci curvature, which were developed more recently. In passing we will mention new results obtained thanks to a discrete viewpoint on Ricci curvature, both in a surprisingly diverse array of discrete settings and in the original setting of Riemannian manifolds .

2.1. **Discrete and non-smooth sectional curvature**: Gromov-hyperbolicity and Alexandrov cur-vature . On their sectional curvature there are at least two notions of metric spaces having bounds. The first one is that of **spaces with curvature bounds in the sense of Alexandrov and** is limited to ge-odesic spaces.

(For curvature bounded above also termed **locally** CAT(k) **spaces**, and for curvature bounded below termed as $CAT^{+}(k)$ **spaces locally**). The second is applies to any metric space that of δ**-hyperbolic spaces** (or **Gromov-hyperbolic spaces**) but only for negative curvature "at large scale".

In negative curva-ture, in particular both notions have been enormously successful .As δ-hyperbolic spaces and CAT(0) spaces have deep relations to geometric group theory. All the relevant work cannot be listed here and for a more extensive treatment we would like to refer the reader to [BH99] or [BBI01].

In the sense of Alexandrov go curvature bounds are as follows. Considering X as a geodesic space. A **triangle** in X is a represented as triplet of points $(a, b, c) \in X^3$, organized with three curves from a to b, b to c and c to a, respectively (which represents the **sides** of the triangle), in such a way that the lengths of these curves realize the distances d(a, b), d(b, c) and d(c, a), respectively.

CAT criterion, for Cartan– Alexandrov–Topogonov also termed as the curvature criterion of Alexandrov , states that in negative cur-vature triangles become "thinner" rather in positive curvature it becomes "fatter". This will be measured by The distance between a vertex of the triangle and a point on the opposite side is used to measure this fact, related to that expected in the Euclidean situation.

Considering v be a unit tangent vector at some point x in the Riemannian manifold X.and let C be any small neighborhood of X, having an arbitrary shape. for each point

.Let X be a geodesic space and k a real number. X is a **space of curvature** 6 k **in the sense of Alexandrov** if, for any small enough triangle (a, b, c) in X, and for any point x on the side bc of this triangle, the following holds:

$$d(a, x)\ 6\ d_{S\kappa}(A_k, X_k)$$

where (A_k, B_k, C_k) is the comparison triangle of (a, b, c) in curvature k, and X_k is the point on the side B_kC_k corresponding to x, i.e. such that $d(X_k, B_k) = d(x, b)$. Similarly, X is a **space of curvature** > k **in the sense of Alexandrov** if, in the same situation, the reverse inequality

$$d(a, x) > d_{S\kappa}(A_k, X_k)$$

holds.

With the Riemannian notion of sectional curvature this definition is compatible . For instance, we have the following ([BH99], Theorem II.1A.6).

ISSN (O) 2454-9665
ISSN (P) 2454-0663

THEOREM 2.2. **Let a complete smooth Riemannian manifold represented by X and** k ∈ R**. Then in the sense of Alexandrov** X **has curvature at most** k **if and only if for any two tangent vectors** v, w **at the same point, with unit length and orthogonal to each other, one has** K(v, w) 6 k**.**

When **all** triangles, not only small ones, satisfy the comparison criterion then the terminology CAT(k) **space**, for curvature 6 k is applied. This is equivalent to negative sectional curvature and being simply connected for complete smooth Riemannian manifolds. For spaces of curvature bounded **below**, imposing the condition on small triangles implies the same for all triangles (Topogonov's theorem, Theorem 10.5 in [BBI01]).

A number of properties of Riemannian manifolds with controlled sectional curvature is kept with the spaces with curvature bounds in the sense of Alexandrov. For instance, in this context states that any CAT(0) space is contractible by the Cartan–Hadamard theorem (Theorem 1.5) ([BH99], Corollary II.1.5).

δ-hyperbolicity (also called Gromov-hyperbolicity) is a very successful notion of negative sectional cur-vature for **discrete** spaces ,. This notion was thoroughly developed by Gromov [Gro87] and is usually attributed to Rips and, especially in the context of geometric group theory, where it led to the concept of hyperbolic groups.

At large scales, a space behaves metrically almost like a tree stated by δ-hyperbolicity.At small scales any phenomena occurring is discarded; precisely, a bounded metric space X is always δ-hyperbolic with δ = diam X. Triangle is involved in the simplest definition of δ-hyperbolicity and the space is geodesic is being assumed. The condition is expressed in terms of quadrilaterals, for non-necessarily geodesic metric spaces.

The large n-gons in the hyperbolic plane look almost like trees , their sides clinging to each other is the primary idea. Consider four points w, x, y, z in a tree and the ideal limit case . No branch of the subtree joining these points is degenerate is assumed in the general situation . This subtree is made of five segments, one of which lies in the middle.

Let us consider d(w, x) + d(y, z), d(w, y) + d(x, z) and d(w, z) + d(x, y) the three pairwise sums of distances. The smallest one does not involve the length of the middle segment of the tree of these three sums, the other two are equal.The ultrametric-like inequality can summed up by

$$d(w, x) + d(y, z) \ 6 \ \max (d(w, y) + d(x, z), d(w, z) + d(x, y))$$

for all points w, x, y, z in a metric tree. Among the three sums involved, the largest two must be equal is what we get by permuting the roles of the points (the reason behind is the largest is bounded by the max of the smallest and second-largest).

We discompose this inequality by an additive term.The metric relations between any four points are the same as in a tree up to a small error in a hyperbolic space.

DEFINITION 2.3. Let δ > 0. A metric space X is δ-**hyperbolic** if

$$d(w, x) + d(y, z) \ 6 \ \max (d(w, y) + d(x, z), d(w, z) + d(x, y)) + 2\delta$$

for any points w, x, y, z in X.

Any local topological property does not imply this when δ > 0.At scales smaller than δ nothing is predictable.This definition is its invariance under **quasi-isometries** is one of the key feature [BH99].

Any Riemannian manifold with negative sectional curvature and which is simply connected, is δ-hyperbolic and in this sense this notion is an extension of negative sectional curvature "at large scales".δ-hyperbolicity is compatible with global negative curvature in the sense of Alexandrov since CAT(−k) spaces for k > 0 are δ-hyperbolic δ-

ISSN (O) 2454-9665
ISSN (P) 2454-0663

hyperbolic groups (groups whose Cayley graph is δ-hyperbolic) keep many of the features of fundamental groups of negatively and are important objects and curved manifolds. Among discrete groups these groups turn out to be, in some sense, "generic" [Gro87, Oll05].

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